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LETTER TO THE EDITOR

General position-space renormalisation group for correlated percolation

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Abstract. A position-space renormalisation group (PSRG) approach has been developed for general site percolation problems in which the site occupancies are correlated, rather than independent, random variables. PSRG parameters are systematically introduced to describe multiple-site correlations. At the two- and four-site levels of approximation in the square lattice we find only two physical fixed points. One characterises random, or pure, percolation and has a universality class which is shown to encompass a variety of locally correlated site problems, including unfrustrated plaquettes and four-coordinated sites in a random-bond model. The second describes 'Ising-correlated' percolation and at the two-site level yields an excellent estimate (to within 2%) for the nearest-neighbour spin correlation function of the Ising model.

Percolation problems on a lattice typically involve the large-scale connectivity properties of a network of sites or bonds which are independent random variables. That is, the likelihood that an element (site or bond) is present is independent of that of its neighbours. A large number of connectivity problems, however, do not share this property, in that the probability of occupancy may depend on the presence or absence of neighbouring elements.

Such 'correlated percolation' problems may arise for example in the context of the Ising model, if spins of one sign are regarded as occupied sites, and those of the opposite sign as empty. In this case the site correlation is generated by the interaction term in the Hamiltonian of the system. Since the correlation is a function of temperature we may refer to it as 'thermally induced'. Correlated percolation in the Ising model has been studied recently by Coniglio *et al* (1977), Klein *et al* (1978), Murata (1979) and Coniglio and Klein (1980).

In other situations involving correlated sites or bonds, however, a Hamiltonian formulation may be inappropriate. Systems involving quenched disorder may develop correlations which are 'geometrically induced'. We consider, for instance, site percolation in a system for which the site occupancies are determined by a quenched random property of the dual lattice. Specific examples include the percolation of four-coordinated sites (that is, sites with four bonds emerging from them) on a square lattice in which the bonds are empty or occupied at random (Stanley 1979, Gonzalez and Reynolds 1980, Blumberg *et al* 1980). A similar problem involves the percolation

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of unfrustrated plaquettes in the $\pm J$ random-bond Ising model (Tuthill 1981) where a plaquette (or elementary loop) of the lattice is defined as frustrated if the number of negative bonds in its perimeter is odd. In both of these examples the presence or absence of a site (that is, four-coordinated site or unfrustrated plaquette) is not independent of the condition of a neighbouring site, but the 'interaction' cannot be simply described by parameters in a global statistical weight, or Boltzmann factor. Approximation methods which rely on the existence of a Hamiltonian to incorporate correlation properties are thus ill suited to studying the latter class of problems. What is necessary is a general technique which focuses only on the local configuration of the network.

In this letter we describe the development of such a general PSRG for correlated percolation, and apply the method to the case of site percolation on the square lattice. The interdependence of site occupancies is taken into account by introducing, in a systematic way, renormalisation parameters which describe multiple-site correlations. The two levels of approximation to be discussed here treat exactly the correlations (a) between two nearest-neighbour sites, and (b) among four sites in a square array. At the two-site level it is sufficient to introduce a single parameter in addition to the net concentration of occupied sites, while in the four-site approximation we must deal with a five-parameter PSRG.

The results for both cases display *only two* physical fixed points. The first dominates a universality class that includes uncorrelated ('pure') percolation, as well as a variety of problems with geometrically induced correlation, including the four-coordinated sites and the unfrustrated plaquettes mentioned above. These problems share the property that the length which characterises the correlation between sites remains finite, and therefore is 'renormalised away' under the rescaling of the PSRG. The second fixed point controls percolation properties in cases of stronger and more long-ranged correlation. Included in this category is the percolation of thermally correlated sites at the critical point of the Ising model. It is important to point out that systems (such as the Baxter model) in which a marginal operator plays an important role cannot be studied by a renormalisation group containing the kind of approximation which we use here. Consequently our conclusion of two universality classes does not include such systems.

Two-site approximation

At the two-site level we must allow for independent probabilities for each of the possible states of a pair of nearest-neighbour sites. Denoting empty and occupied sites by ○'s and ×'s, and assuming lattice isotropy, we define the probabilities u_1, u_2, u_3 by

$$u_1 = \langle \bigcirc\bigcirc \rangle \qquad u_2 = \langle \bigcirc\times \rangle = \langle \times\bigcirc \rangle \qquad u_3 = \langle \times\times \rangle \tag{1}$$

where the angular brackets signify the likelihood of the enclosed configuration. The sum of the probabilities of all states of a particular pair is normalised to one:

$$u_1 + 2u_2 + u_3 = 1. \tag{2}$$

It is convenient to introduce a second set of parameters which are closer to the usual language of percolation. The net concentrations of occupied sites p or empty sites q can be written as

$$p = u_2 + u_3 \qquad q = 1 - p = u_1 + u_2. \tag{3}$$

A single correlation parameter t is then defined by the relation

$$tpq = u_2. \tag{4}$$

Combining equations (2)–(4), we find u_1 and u_3 in terms of p, q and t:

$$u_1 = q(1-tp) \qquad u_3 = p(1-tq). \tag{5}$$

The case of no correlation, in which the site occupancies are independent random variables, amounts to fixing $t = 1$. Positive correlation, or occupied sites tending to cluster together, corresponds to values of t less than one, and negative correlation (or site repulsion) to t greater than one. To ensure positivity of the $\{u_i\}$, we require that

$$0 \leqslant t \leqslant 1/\max(p, 1-p). \tag{6}$$

Strictly speaking, the definition of t (equation (3)) is not meaningful in the limits $p \to 0$ or 1 (all sites empty or occupied). Our basic PSRG equations, however, may be cast in terms of the $\{u_i\}$ whose definitions are straightforward.

To develop a renormalisation procedure we consider a cluster consisting of two adjacent four-site cells, as shown in figure 1. In the rescaled system this will be mapped to two nearest-neighbour sites. A renormalised site is said to be occupied if the cell which it represents contains a spanning cluster, independent of the configuration (occupied or empty) of the neighbouring renormalised site. Specifically

$$p' = \sum' (\text{configuration weights}) \tag{7}$$

where Σ' represents a sum with the constraint that the (say) right cell has a spanning cluster. Clearly this should reduce to the independent cell approximation of Reynolds *et al* (1977, 1978) for $t = 1$. The second PSRG equation is generated by the relation

$$t'p'(1-p') = \sum'' (\text{configuration weights}), \tag{8}$$

Physically the left side of (8) is $\langle \bigcirc\times \rangle$ in the renormalised lattice, so that the sum Σ'' is restricted to those configurations in which the right cell has a spanning cluster and

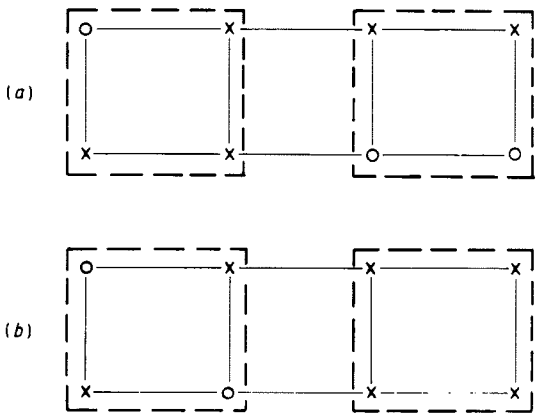

Figure 1. Typical clusters of eight sites used in the PSRG for correlated percolation at the two-site level of approximation. Occupied sites are indicated by crosses, empty sites by open circles. Four-site cells enclosed by broken lines are mapped to sites of the renormalised lattice. Associated with configuration (*a*) is a relative (unnormalised) weight given by expression (9*a*) of the text; the weight of configuration (*b*), which contains a closed loop of occupied sites, is given by (9*b*).

the left has not, as in figure 1(b). We require of course that this yield $t' = 1$ when the sites are uncorrelated—that is, when $t = 1$.

In order to complete our PSRG we must express the weights of the configurations of the eight sites in terms of two-site correlations. We do so by adopting a modified form of the superposition approximation, well known in the theory of fluids (Münster 1969). Specifically, we apply a factor of p or q for each occupied or empty site, and a factor of u_1/q^2, u_2/pq or u_3/p^2 for each bond.

The superposition approximation in fluids is known to be quite good for high temperatures and low densities. At high densities and low temperatures it is known to be a poor approximation—a fact reflected in our PSRG by divergences in the recursion relations as p or q approaches zero when $t < 1$. Roughly speaking, this breakdown arises because the superposition approximation overestimates the correlation in any closed loop. To compensate, we insert a factor of $(u_3/p^2)^{-1}$ for each plaquette consisting only of occupied sites, and a factor of $[u_1/q^2]^{-1}$ for each plaquette made up only of empty sites. This modification is necessary to produce physically reasonable behaviour of the PSRG near $p = 0$ and $q = 0$; however the recursion relations are still not to be taken seriously in those limits. As examples we write the weight for the configuration of figure 1(a) as

$$p^5q^5(u_1/q^2)(u_2/pq)^5(u_3/p^2)^4 = p^2q^2t^5(1-tp)(1-tq)^4 \tag{9a}$$

and that of figure 1(b) as

$$p^6q^2(u_2/pq)^5(u_3/p^2)^5(p^2/u_3) = p^2q^2t^5(1-tq)^4. \tag{9b}$$

The weight formed using our approximation satisfies the following criteria. (a) It correctly reduces to the product of independent site probabilities in the limit $t = 1$. (b) It is non-singular in the $p, q = 1$ limits when $t = 1$; only the configuration with all sites occupied (or empty, respectively) has non-vanishing probability. (c) In the $t = 0$ limit, the likelihood of finding a cluster with all sites occupied is simply p (or the likelihood of finding a single site of that cluster occupied), and that for all sites empty, q. Partially occupied configurations have vanishing probability when $t = 0$.

The resulting RG flows are shown in figure 2 for the case of an R_1 unidirectional connectivity rule under which the cell must be spanned horizontally—that is, in a direction parallel to the axis connecting the two cells. Other connectivity rules yield qualitatively similar flow diagrams. We have plotted our results on a composition triangle with vertices u_1, u_3 and $2u_2$, to avoid confusion arising from the definition of t in the limits of small p or q. In this representation the line of zero correlation ($t = 1$) appears as the curve $u_1 = (u_1 + u_2)^2$, shown as a broken curve in figure 2.

Disregarding the fully stable vertices u_1 and u_3, there are two physically meaningful fixed points. The point R, located at ($u_{1R} = 0.146$, $u_{2R} = 0.236$, $u_{3R} = 0.382$) or ($t_R = 1$, $p_R = 0.618$) corresponds to the percolation threshold for randomly occupied sites. The correlation parameter t acts as an irrelevant scaling field (eigenvalue = 0.215), while for $t = 1$ the recursion relations reduce to those of Reynolds *et al* (1977, 1978). The relevant eigenvalue at R is 1.528, giving a connectedness length exponent for percolation ν_p of 1.635. The fixed point R dominates a universality class which includes the percolation problems with the geometrically induced correlation mentioned above. Plotted in figure 2 are the $u_1(u_2)$ or $t(p)$ lines for both the four-coordinated and the unfrustrated plaquette percolation problems. In this approximation the percolation thresholds for both problems are only slightly reduced from those of the random-site problem, to 0.603 (four-coordinated sites) and 0.614 (unfrustrated

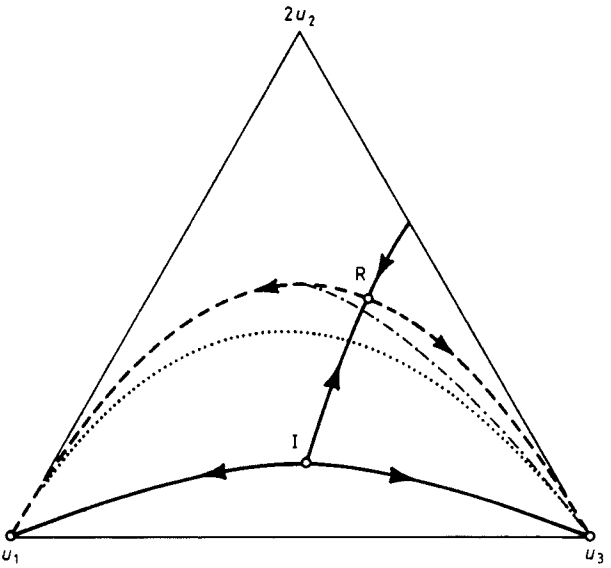

Figure 2. PSRG flow diagram for the two-site approximation. The fixed point R, lying on the broken line of no correlation ($t = 1$), controls percolation exponents along a critical line which terminates at I, the Ising fixed point. Shown also are the lines $u_1(u_2)$ for four-coordinated sites (dotted) and for unfrustrated plaquettes (chain curve). The fraction of unfrustrated plaquettes cannot be less than $\frac{1}{2}$, so that u_1 is restricted to values $<\frac{1}{4}$.

plaquettes). The per cent reductions from the random-site percolation thresholds are 2.4% and 0.6% respectively, to be compared with the values 5.3% (Blumberg *et al* 1980) and <0.5% (Tuthill 1981) found from Monte Carlo studies.

The remaining non-trivial fixed point, labelled I in figure 2, is located at ($u_{1I} = 0.417$, $u_{2I} = 0.726$, $u_{3I} = 0.438$) or ($t_I = 0.291$, $p_I = 0.511$). There are two relevant scaling fields at this point—approximately pure deviations in p and t from their fixed point values—with the respective scaling powers

$$y_p = 0.629 \qquad y_t = 0.981. \tag{10}$$

We refer to this as the Ising percolation fixed point, since the thermal critical point of the two-dimensional Ising model is known to be a higher-order critical point for the percolation of spin-up (or spin-down) sites. That is, it is the terminus of a line of percolation points in magnetic field–temperature space (Coniglio *et al* 1977, Klein *et al* 1978, Coniglio and Klein 1980).

In this regard it is especially interesting that p_I and t_I agree remarkably well with the exact critical values p_c and t_c for the two-dimensional Ising ferromagnet. For spins s_{ij} which can take on the values ± 1, the spin–spin correlation function for nearest neighbours can be expressed as

$$\langle s_{ij} s_{ij+1} \rangle = 1 - 4tpq. \tag{11}$$

The left-hand side of equation (11) is known to take the value $2^{-1/2}$ at the critical point (Kaufman 1949, Kaufman and Onsager 1949, Montroll *et al* 1963). Using the fact that p is $\frac{1}{2}$ at zero field and the critical temperature, we obtain the Ising critical point values

$$p_c = 0.5 \qquad t_c = 1 - 2^{-1/2} = 0.2929. \tag{12}$$

The scaling powers at the point I may also be compared with those of the Ising model. The parameter t is expected to scale as the two-spin correlation function—that is, as $(T-T_c)^{1-\alpha}$ where T_c is the critical temperature and $\alpha = 0$ for the two-dimensional Ising system. We therefore expect $y_t = y_T$ (Ising) $= 1.00$, which is well approximated by our result. Our calculated y_p should serve as an approximation for the Ising magnetisation scaling power, $y_M = 0.125$. The poor agreement reflects the somewhat arbitrary nature of the 'closed loop' correction introduced into our superposition approximation; y_p is extremely sensitive to the exact form of this correction, while y_t is not.

The part of the PSRG flow diagram for small u_2 corresponds to a region of 'phase separation', in which is favoured the formation of compact macroscopic clusters (with boundaries $\ll$ volumes) of occupied or empty sites. Due to criterion (c) above, $u_2 = 0$ is a fixed line.

Four-site approximation

There are six distinct configurations (modulo rotations) of the square array of four lattice sites. We therefore define the probabilities w_1 to w_6 by

$$w_1 = \langle \begin{smallmatrix} \circ\circ \\ \circ\circ \end{smallmatrix} \rangle \qquad w_2 = \langle \begin{smallmatrix} \circ\circ \\ \circ\times \end{smallmatrix} \rangle = \langle \begin{smallmatrix} \circ\times \\ \circ\circ \end{smallmatrix} \rangle \text{ etc} \qquad w_3 = \langle \begin{smallmatrix} \circ\circ \\ \times\times \end{smallmatrix} \rangle = \langle \begin{smallmatrix} \times\circ \\ \times\circ \end{smallmatrix} \rangle \text{ etc}$$

$$w_4 = \langle \begin{smallmatrix} \circ\times \\ \times\circ \end{smallmatrix} \rangle = \langle \begin{smallmatrix} \times\circ \\ \circ\times \end{smallmatrix} \rangle \qquad w_5 = \langle \begin{smallmatrix} \times\times \\ \times\circ \end{smallmatrix} \rangle = \langle \begin{smallmatrix} \times\circ \\ \times\times \end{smallmatrix} \rangle \text{ etc} \qquad w_6 = \langle \begin{smallmatrix} \times\times \\ \times\times \end{smallmatrix} \rangle. \tag{13}$$

Normalisation implies the condition

$$w_1 + 4w_2 + w_3 + 2w_4 + 4w_5 + w_6 = 1. \tag{14}$$

We note that the two-site parameters are easily expressed in terms of the w's:

$$u_1 = w_1 + 2w_2 + w_3 \qquad u_2 = w_2 + w_3 + w_4 + w_5 \qquad u_3 = w_3 + 2w_5 + w_6$$

or

$$p = w_2 + 2w_3 + w_4 + 3w_5 + w_6 \qquad tp(1-p) = w_2 + w_3 + w_4 + w_5. \tag{15}$$

For spatial rescaling by a factor of two, we use a square cluster of sixteen lattice sites (figure 3). Individual configurations again receive statistical weights according to a superposition approximation, this time involving factors of w_i for each four-site

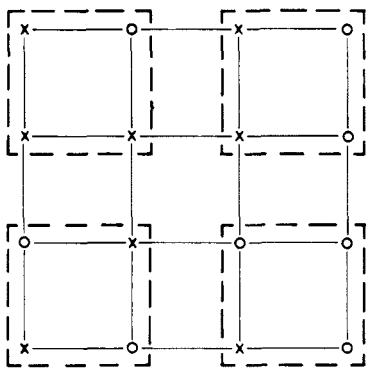

Figure 3. Typical cluster of sixteen sites used in the four-site approximation. This configuration would be assigned a relative weight $w_2^2 w_3 w_4^2 w_5^3 u_1^{-1} u_2^{-5} u_3^{-2}$.

cell (excepting the centre cell) and $1/u_j$ for each bond shared between these cells. To decide whether or not a renormalised site is occupied we apply an R_1-type connectivity rule (modified slightly to preserve lattice isotropy, in that cells of type w_3 are considered spanned if the occupied sites are shared with a neighbouring cell).

Two physical fixed points are again found, one (R) which controls random or pure percolation and is stable against weak correlation, and a second (I) at strong positive correlation. These are located at the positions shown in table 1, where we list as well the respective p and t for each fixed point. Listed also in table 1 are the relevant eigenvalues λ_i at each point.

Table 1. Fixed point locations and relevant eigenvalues in the four-site approximation.

Fixed point	R (pure percolation)	I (positive correlation)
w_1	0.02129	0.14282
w_2	0.03444	0.03456
w_3	0.05573	0.03851
w_4	0.05573	0.00908
w_5	0.09017	0.05153
w_6	0.14590	0.34065
p	0.61803	0.61588
t	1.0	0.56505
λ_1	1.528	1.400
λ_2	—	1.082

At the point R we again have a single relevant eigenvalue, identical to that found in the two-site approximation. This reflects the fact that our superposition approximation correctly reproduces the R_1 recursion relations for the case of uncorrelated sites.

The percolation points for four-coordinated sites and for frustrated plaquettes are found to be

$$p \text{ (four-coordinated sites)} = 0.602 \qquad p \text{ (unfrustrated plaquettes)} = 0.615 \tag{16}$$

in good agreement with the results of the two-site approximation.

The location of the point I now fails to provide a good estimate of the Ising critical point, but its basic character remains the same as in the two-site approximation. In particular, we find two relevant fields, with scaling powers $y_{1,2} = 0.485, 0.1144$. A four-spin correlation in the two-dimensional Ising system should behave near the critical point like $(T - T_c)^{-\alpha}$; since $\alpha = 0$, the existence of one scaling power near zero in our results is perhaps not surprising. As before, the point is unstable with respect to perturbations in the correlation parameter t, with PSRG flows going off in the direction of total correlation ($t = 0$) or toward zero correlation. A separatrix leads from I to the random fixed point R.

We point out that the introduction of the four-site configuration probabilities w_i allows a rough test of the superposition formula which was used for the two-site approximation. Consider namely a four-site cell and specific values for the w's. Using the relation (12) to calculate p and t, we may apply the two-site superposition formulae to estimate the likelihood of each configuration, and compare with the original w_i. Even at the fixed point I, where the site correlation is strong and superposition

therefore less reliable, we find deviations from the correct values averaging only about 13%.

To summarise, we have developed a general PSRG approach for site percolation problems in which the sites are arbitrarily correlated random variables. Our approach does not depend on the existence of a Hamiltonian to establish the correlation, but instead uses a superposition approximation to estimate the likelihoods of multi-site configurations. We have applied this PSRG to the two-dimensional square lattice, at the two-site and four-site levels of approximation, and find only two non-trivial fixed points—for pure random and for Ising-correlated percolation. We suggest that these in fact characterise the only two universality classes for two-dimensional percolation problems (in which marginal operators are absent), distinguishing systems with finite site-correlation lengths from those in which such lengths diverge. Further extensions of this work, in particular to three-dimensional systems, will be described in a subsequent publication.

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